Supplementary Midterm Examination
Classical Mechanics

B. Math., 2^{nd} Year, July - November 2024,
Date and Time: October 10, 2024, 10 a.m,
Duration: 2 hours,
Total points: 50

You must provide clear argument(s) for all steps for full grade. An answer with zero explanation, even if correct, will not receive any credit.

1. Consider the double pendulum: it consists of two massless inextensible rods, with one of the rods (the top rod; length l_1) fixed to a stationery fulcrum in space. The other rod (the bottom rod; length l_2) is affixed to the bottom of the top rod (this point is the join). At the bottom of the top rod is a particle of mass m_1 and at the bottom of the bottom rod is a mass m_2. Consider the motion of the system in a single vertical plane under gravity. Write down the Lagrangian and derive the equation of motion of the system, using the generalized co-ordinates θ_1 and θ_2. Here, θ_1 is the instantaneous angle between the downward vertical and the top rod as measured at the fulcrum, while θ_2 is the instantaneous angle between the downward vertical and the bottom rod as measured from the join.

   10 points

2. Suppose the Lagrangian of a system L is modified by adding a total time -derivative: i.e.,
$$L_{\text{new}} = L + \frac{dF}{dt}$$
   where F is an arbitrary but differentiable function of the generalized co-ordinates and the generalized velocities. Show that L_{new} satisfies the same Euler-Lagrange equations as L.

   10 points

3. A pendulum consists of a point mass m fixed to the bottom of a massless stick of length l. The top of the stick is fixed to a horizontal support which itself moves horizontally such that its position follows $X(t) = A\cos(\omega t)$.

   (a) Find the equations of motion of the mass through the Lagrangian formulation.

(b) Make a small angle approximation: i.e., imagine that the instantaneous angle that the stick + mass makes with the support is small (include terms up to linear order in the angle). Show that you get the equation of a driven oscillator (i.e., an oscillator/spring with an explicitly time-dependent force other than the spring force).

7 + 8 = 15 points

4. Imagine a particle of mass m moving in a central potential of the form $-\alpha/r^2$, $\alpha > 0$. The value of the angular momentum is fixed at L. In an $E - r$ diagram, analyze the motion for the various values of total energy E and α qualitatively.

   15 points